The UNNS Many-Faces Theorem: Formalization and Proof Sketches

(with Fibonacci embedding lemma)

Research Note

Abstract

We propose and sketch a formal theorem—the "Many-Faces Theorem"—for the UNNS framework (Unbounded Nested Number Sequences / Universal Network Nexus System). The result shows how linear recurrence sequences, attractors, modular domains, and cross-domain homomorphisms are all naturally embeddable in UNNS. This version adds a concrete lemma proving that the Fibonacci sequence is exactly representable in UNNS, together with a proof of the convergence to the golden ratio.

1 Definitions (abridged)

Definition 1 (UNNS system — abridged)**.** *A UNNS system is a tuple* $U = (S, \mathcal{C}, \mathcal{G}, \{\mu_D\}_{D\in\mathcal{D}})$ *where:*

- S *is a set of nests (here we may take* $S \subseteq \mathbb{Z}$ *or symbolic descriptions),*
- $\mathcal{C}$ *is a finite set of combinators* $\star : S^k \to S$ *(finite algorithms using integer arithmetic and bounded lookback),*
- $\mathcal{G} \subset S$ *is a finite seed set,*
- *each* μ_D *is a computable mapping into a domain* D *(e.g.,* $\mathbb{Z}$*,* $\mathbb{R}^2$*,* $\mathbb{Z}_m$*).*

2 Main theorem (recap)

(See the UNNS Many-Faces Theorem as stated previously: linear recurrence embedding, dominant-root attractor, modular partition, cross-domain homomorphism, conditional computational completeness.)

We now add a concrete, fully formalized lemma for the Fibonacci case.

Lemma 1 (Fibonacci embedding in UNNS)**.** *There exists a UNNS system* $U = (S, \mathcal{C}, \mathcal{G}, \{\mu_D\})$ *satisfying the UNNS assumptions such that the sequence of nest values* $\{s_n\}_{n\geq 0} \subset S$ *produced by iterated application of a single combinator* $\star \in \mathcal{C}$ *satisfies*

$$\mu_{\mathbb{Z}}(s_n) = F_n \quad \text{for all } n \geq 0,$$

where F_n *is the Fibonacci sequence defined by* $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$*. Furthermore,*

$$\lim_{n\to\infty} \frac{\mu_{\mathbb{R}}(s_{n+1})}{\mu_{\mathbb{R}}(s_n)} = \varphi,$$

the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$*.*

Proof. We give a constructive UNNS encoding and prove correctness by induction; then we prove the convergence result.

Construction (UNNS instance). Define:

- $S = \mathbb{Z}$ (nests carry integer values).
- $\mathcal{G} = \{s_0, s_1\}$ with $s_0 := 0, s_1 := 1$.
- $\mathcal{C} = \{\star\}$ where $\star$ is a binary combinator operating on the most recent two nests: given the ordered pair (x, y) (with y newest, x previous), define

$$\star(x, y) := x + y.$$

  Operationally, the UNNS update rule maintains a sliding window of the last two nests; at each step it sets the next nest to $\star(s_{n-1}, s_n)$.

- Domain mapping $\mu_{\mathbb{Z}} : S \to \mathbb{Z}$ is the identity map (interpreting each nest as its integer value). Similarly $\mu_{\mathbb{R}}$ is the inclusion into $\mathbb{R}$.

This construction respects the UNNS assumptions: $\star$ is a finite algorithm using integer addition and bounded lookback (arity 2).

Inductive correctness. Define the UNNS-generated sequence $\{s_n\}$ by $s_0 = 0$, $s_1 = 1$, and for all $n \geq 1$,

$$s_{n+1} := \star(s_{n-1}, s_n) = s_{n-1} + s_n.$$

We show by induction on n that $\mu_{\mathbb{Z}}(s_n) = F_n$ for all n.

Base cases: $n = 0$ and $n = 1$ hold by construction: $s_0 = 0 = F_0$, $s_1 = 1 = F_1$.

Inductive step: Suppose for some $n \geq 1$ that $\mu_{\mathbb{Z}}(s_k) = F_k$ for all $k \leq n$. Then

$$\mu_{\mathbb{Z}}(s_{n+1}) = \mu_{\mathbb{Z}}(\star(s_{n-1}, s_n)) = \mu_{\mathbb{Z}}(s_{n-1} + s_n) = \mu_{\mathbb{Z}}(s_{n-1}) + \mu_{\mathbb{Z}}(s_n) = F_{n-1} + F_n = F_{n+1}.$$

Thus the property holds for $n + 1$. By induction, $\mu_{\mathbb{Z}}(s_n) = F_n$ for every $n \geq 0$.

Convergence to φ. The Fibonacci recurrence's characteristic polynomial is $r^2 - r - 1 = 0$ with roots

$$\varphi = \frac{1 + \sqrt{5}}{2}, \qquad \psi = \frac{1 - \sqrt{5}}{2}.$$

Binet's formula states (and can be proved by linear algebraic methods)

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

Therefore

$$\frac{F_{n+1}}{F_n} = \frac{\varphi^{n+1} - \psi^{n+1}}{\varphi^n - \psi^n} = \frac{\varphi - \psi(\psi/\varphi)^n}{1 - (\psi/\varphi)^n}.$$

Since $|\psi| < 1$ and $|\psi/\varphi| < 1$, $(\psi/\varphi)^n \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi.$$

Because $\mu_{\mathbb{R}}$ is inclusion, $\mu_{\mathbb{R}}(s_n) = F_n$ as real numbers, hence the same limit holds for the UNNS-generated nests:

$$\lim_{n\to\infty} \frac{\mu_{\mathbb{R}}(s_{n+1})}{\mu_{\mathbb{R}}(s_n)} = \varphi.$$

Geometric attractor remark. If we define a geometric embedding $\mu_G(s_n) = (\theta_n, r_n)$ with $\theta_n = \gamma n$ and $r_n = \alpha F_n$ for fixed positive constants α, γ, then asymptotically

$$\frac{r_{n+1}}{r_n} = \frac{F_{n+1}}{F_n} \to \varphi,$$

so the polar radii grow geometrically with factor φ and the plotted points approach a logarithmic spiral of form $r(\theta) \approx Ce^{(\ln \varphi/\gamma)\theta}$. This makes φ a geometric attractor under μ_G. □

Corollary 1. *Under the UNNS construction above, the Fibonacci embedding is an instance of Part 1 and Part 2 of the UNNS Many-Faces Theorem: (i) exact recurrence embedding; (ii) dominant-root attractor with φ.*

3 Discussion and next steps

The Fibonacci lemma above serves as a canonical worked example: it demonstrates the constructive embedding technique and the route from recurrence rules to attractor geometry. Similar lemmas follow for other linear recurrences (Pell, Tribonacci, Padovan) by adapting the combinator coefficients and seed values.

For fuller formalization one can:

- Formalize a small combinator language for UNNS and prove general embedding lemmas for arbitrary linear constant-coefficient recurrences (the constructive method used in the Fibonacci lemma generalizes directly).
- Provide formal statements for geometric embeddings (choice of γ, α) and rigorous asymptotic bounds for finite precision implementations.
- Encode and verify small automatic proofs (e.g., using a proof assistant) for the inductive correctness of combinator constructions.