The UNNS Many-Faces Theorem: Formalization and Proof Sketches

(with Fibonacci, Pell, Tribonacci, Padovan lemmas)

Research Note

Abstract

We present formal lemmas embedding classical linear recurrence sequences (Fibonacci, Pell, Tribonacci, Padovan) into the UNNS framework. Each lemma gives a constructive UNNS encoding (combinator and seeds), an inductive correctness proof that the UNNS nests reproduce the sequence, and a short argument that the ratio of consecutive terms converges to the dominant root of the characteristic polynomial (hence a geometric attractor under a polar embedding).

1 Definitions (abridged)

Definition 1 (UNNS system — abridged)**.** *A UNNS system is a tuple* $U = (S, \mathcal{C}, \mathcal{G}, \{\mu_D\}_{D\in\mathcal{D}})$ *where:*

- S *is a set of nests (here we may take* $S \subseteq \mathbb{Z}$ *or symbolic descriptions),*
- $\mathcal{C}$ *is a finite set of combinators* $\star : S^k \to S$ *(finite algorithms using integer arithmetic and bounded lookback),*
- $\mathcal{G} \subset S$ *is a finite seed set,*
- *each* μ_D *is a computable mapping into a domain* D *(e.g.,* $\mathbb{Z}$*,* $\mathbb{R}^2$*,* $\mathbb{Z}_m$*).*

2 Main theorem (recap)

(See the UNNS Many-Faces Theorem as stated previously: linear recurrence embedding, dominant-root attractor, modular partition, cross-domain homomorphism, conditional computational completeness.)

The following lemmas give canonical constructive embeddings for several classical recurrences.

3 Fibonacci lemma (recap)

Lemma 1 (Fibonacci embedding in UNNS)**.** *There exists a UNNS system* $U = (S, \mathcal{C}, \mathcal{G}, \{\mu_D\})$ *satisfying the UNNS assumptions such that the sequence of nest values* $\{s_n\}_{n\geq 0} \subset S$ *produced by iterated application of a single combinator* $\star \in \mathcal{C}$ *satisfies*

$$\mu_{\mathbb{Z}}(s_n) = F_n \quad \text{for all } n \geq 0,$$

where F_n *is the Fibonacci sequence* $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$*. Furthermore,*

$$\lim_{n\to\infty} \frac{\mu_{\mathbb{R}}(s_{n+1})}{\mu_{\mathbb{R}}(s_n)} = \varphi = \tfrac{1+\sqrt{5}}{2}.$$

Proof. (See the constructive encoding and induction proof in the previous version. Binet's formula gives convergence to φ.) $\square$

4 Pell lemma

Lemma 2 (Pell embedding in UNNS)**.** *There exists a UNNS system $U = (S, \mathcal{C}, \mathcal{G}, \{\mu_D\})$ such that the UNNS-generated sequence $\{p_n\}$ satisfies*

$$\mu_{\mathbb{Z}}(p_n) = P_n \quad \text{for all } n \geq 0,$$

where $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ (the Pell sequence). Moreover,

$$\lim_{n \to \infty} \frac{P_{n+1}}{P_n} = 1 + \sqrt{2}.$$

Construction. Take $S = \mathbb{Z}$. Let seeds $\mathcal{G} = \{p_0, p_1\}$ with $p_0 = 0, p_1 = 1$. Define a binary combinator $\star \in \mathcal{C}$ acting on the last two nests:

$$\star(x, y) := 2y + x.$$

The UNNS update maintains the sliding window (p_{n-1}, p_n) and sets $p_{n+1} = \star(p_{n-1}, p_n) = 2p_n + p_{n-1}$.

Induction. Base cases: $p_0 = 0 = P_0$ and $p_1 = 1 = P_1$. Assume $\mu_{\mathbb{Z}}(p_k) = P_k$ for $k \leq n$. Then

$$\mu_{\mathbb{Z}}(p_{n+1}) = \star(p_{n-1}, p_n) = 2p_n + p_{n-1} = 2P_n + P_{n-1} = P_{n+1}.$$

Thus the equality holds for all n by induction.

Convergence. The characteristic polynomial is $r^2 - 2r - 1 = 0$ with roots $r = 1 \pm \sqrt{2}$. The dominant root is $1 + \sqrt{2}$. Standard linear recurrence theory (Binet-like decomposition for Pell) yields

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}},$$

and hence $P_{n+1}/P_n \to 1 + \sqrt{2}$ as $n \to \infty$. $\square$

5 Tribonacci lemma

Lemma 3 (Tribonacci embedding in UNNS)**.** *There exists a UNNS system $U = (S, \mathcal{C}, \mathcal{G}, \{\mu_D\})$ such that the UNNS-generated sequence $\{t_n\}$ satisfies*

$$\mu_{\mathbb{Z}}(t_n) = T_n \quad \text{for all } n \geq 0,$$

where $T_0 = 0, T_1 = 0, T_2 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ (the Tribonacci sequence). Moreover, the ratio T_{n+1}/T_n converges to the unique real root $\tau \approx 1.8392867552$ of $r^3 - r^2 - r - 1 = 0$.

Construction. Let $S = \mathbb{Z}$. Seeds: $t_0 = 0, t_1 = 0, t_2 = 1$. Define a ternary combinator $\star$ that takes three most recent nests (x, y, z) (with z newest) and returns

$$\star(x, y, z) := x + y + z.$$

The UNNS update stores last three nests and sets $t_{n+1} = \star(t_{n-2}, t_{n-1}, t_n) = t_n + t_{n-1} + t_{n-2}$.

Induction. Base cases hold by construction. Suppose the equality holds up to n. Then

$$t_{n+1} = t_n + t_{n-1} + t_{n-2} = T_n + T_{n-1} + T_{n-2} = T_{n+1},$$

so induction completes.

Convergence. The characteristic polynomial $r^3 - r^2 - r - 1 = 0$ has one real root $\tau > 1$ and two complex roots of smaller magnitude. By root decomposition, $T_n = A\tau^n + B\rho^n + C\overline{\rho}^n$ where $|\rho| < \tau$. Thus $T_{n+1}/T_n \to \tau$ as $n \to \infty$. □

6 Padovan lemma

Lemma 4 (Padovan embedding in UNNS)**.** *There exists a UNNS system $U = (S, \mathcal{C}, \mathcal{G}, \{\mu_D\})$ such that the UNNS-generated sequence $\{q_n\}$ satisfies*

$$\mu_{\mathbb{Z}}(q_n) = P'_n \quad \textit{for all } n \geq 0,$$

where the Padovan sequence (variant) is defined by initial seeds $P'_0 = 1, P'_1 = 1, P'_2 = 1$ and recurrence $P'_n = P'_{n-2} + P'_{n-3}$. Moreover, the ratio P'_{n+1}/P'_n converges to the unique real root $\rho \approx 1.3247179572$ of $r^3 - r - 1 = 0$ (the plastic constant).

Construction. As before, take $S = \mathbb{Z}$. Seeds: $q_0 = 1, q_1 = 1, q_2 = 1$. Define a ternary combinator $\star$ (or alternately a bounded lookback combinator that uses indices $n-2, n-3$) by

$$\star(x, y, z) := x + y \quad \text{(when applied with the appropriate ordering to yield } q_n = q_{n-2} + q_{n-3}\text{)}.$$

Operationally one implements the sliding index so that at step n the combinator computes $q_n = q_{n-2} + q_{n-3}$ from the stored 3–4 previous values. This is allowed since combinators have bounded lookback by assumption.

Induction. Base seeds match. Inductively, suppose the sequence matches for indices up to n. Then the combinator produces

$$q_{n+1} = q_{n-1} + q_{n-2} = P'_{n+1},$$

so the equality holds for all n.

Convergence. Padovan's characteristic polynomial $r^3 - r - 1 = 0$ has a unique real root $\rho > 1$ (the plastic constant) and two complex conjugate roots of smaller magnitude. Standard linear recurrence decomposition gives $P'_n = A\rho^n + B\alpha^n + C\overline{\alpha}^n$ with $|\alpha| < \rho$. Hence $P'_{n+1}/P'_n \to \rho$ as $n \to \infty$. □

7 Corollaries

Corollary 1. *Each of Fibonacci, Pell, Tribonacci, and Padovan sequences is constructively embeddable into UNNS as shown; thus Parts 1 and 2 of the Many-Faces Theorem are illustrated by concrete examples.*

8 Remarks

Proof. Proof. Proof. The combinators used above are linear (integer coefficients) and have bounded lookback, so they satisfy the UNNS assumptions; generalization to other linear recurrences is straightforward by adjusting coefficients and window size.

- Convergence proofs rely on standard characteristic-polynomial decomposition; in each case the presence of a unique real dominant root suffices to ensure ratio convergence.
- Nonlinear recurrences or recurrences with equal-magnitude dominant roots require separate treatment.

9 Next steps

One may:

1. Formalize a small typed combinator language for UNNS and state a general embedding theorem for arbitrary linear constant-coefficient recurrences.
2. Provide numerical demonstrations (plots of ratios, spiral embeddings) and include those figures via `\includegraphics` for a research article.
3. Optionally encode the combinator constructions into a proof assistant to obtain machine-checked inductive proofs.