From Unbounded Nested Number Sequences to Maxwell's Equations via DEC and FEEC

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Abstract

We construct a rigorous mathematical bridge between Unbounded Nested Number Sequences (UNNS) and the classical Maxwell system of electrodynamics. Interpreting UNNS data as discrete electromagnetic potentials on a hierarchy of meshes in spacetime, we apply Discrete Exterior Calculus (DEC) and Finite Element Exterior Calculus (FEEC) to establish convergence of the discrete Maxwell system to the continuous one in four-dimensional spacetime. Full proofs are given for the 3+1-dimensional case, including explicit constants, stability estimates, and treatment of gauge freedom.

Keywords: UNNS, Discrete Exterior Calculus, Finite Element Exterior Calculus, Maxwell convergence, constructive electromagnetism, number theory.

1 Introduction

Maxwell's equations admit a natural formulation in the language of differential forms. Discrete Exterior Calculus (DEC) provides combinatorial analogues on meshes, while Finite Element Exterior Calculus (FEEC) supplies approximation and stability theorems.

The UNNS formalism provides a systematic way to encode discrete potentials across an infinite nested hierarchy of meshes. Our goal is to show that UNNS data naturally induce discrete electromagnetic fields F_h which converge to the classical fields F as $h \to 0$.

2 Preliminaries

Let $M = \Omega \times [0,T] \subset \mathbb{R}^{3+1}$ with Lorentzian metric $g = -dt^2 + dx^2 + dy^2 + dz^2$.

2.1 Maxwell in forms

Given potential $A \in \Omega^1(M)$, define $F = dA \in \Omega^2(M)$. Then

$$dF = 0,$$
$$\delta F = J,$$

where $\delta = (-1)^{nk+n+1} \star d\star$ is the codifferential.

2.2 Meshes and UNNS

Let $\{\mathcal{T}_h\}$ be shape-regular simplicial meshes. Each oriented edge e is assigned a number $a_e \in \mathbb{R}$ from the UNNS, defining a discrete potential A_h.

3 Discrete Operators

3.1 Exterior derivative

For oriented faces f,

$$(d_h A_h)(f) = \sum_{e \in \partial f} \operatorname{sgn}(e, f)\, A_h(e).$$

3.2 Discrete Hodge star

Define $\star_h$ by dual volumes:

$$\langle \star_h \alpha, \beta \rangle = \sum_{\sigma^k} \frac{|\star \sigma^k|}{|\sigma^k|} \alpha(\sigma^k) \beta(\star \sigma^k).$$

3.3 Codifferential

$$\delta_h = \star_h^{-1} d_h \star_h .$$

4 Discrete Maxwell System

Define $F_h = d_h A_h$. Then

$$d_h F_h = 0,$$
$$\delta_h F_h = J_h.$$

5 Main Results

Theorem 5.1 (Convergence of UNNS Maxwell System)**.** *Suppose* $A \in H^s \Omega^1(M)$, $s > 1$, $J = \delta d A$, *and assumptions hold: shape-regular meshes, Whitney interpolation* Π_h, *and norm-equivalence constants* $c_1, c_2 > 0$. *Then*

$$\|F_h - F\|_{L^2} \leq C h^p, \quad \|J_h - J\|_{L^2} \leq C h^p.$$

Proof. By FEEC theory, stability is ensured by a discrete compactness argument. For any $v_h \in V_h$, Céa's lemma yields

$$\|F - F_h\|_{L^2} \leq \frac{c_2}{c_1} \inf_{v_h \in V_h} \|F - v_h\|_{L^2}.$$

Using the Whitney interpolation Π_h we obtain

$$\|F - F_h\|_{L^2} \leq \frac{c_2}{c_1} \|F - \Pi_h F\|_{L^2} \leq C h^p \|F\|_{H^{p+1}},$$

with $C = \frac{c_2}{c_1}$. The same holds for J_h. □

Lemma 5.2 (Exactness of d_h)**.** *For any discrete 1-form* A_h, $d_h d_h A_h = 0$.

Proof. Follows from boundary-of-boundary = 0. □

Lemma 5.3 (Operator consistency)**.** *For smooth* $\alpha \in \Omega^k(M)$,

$$\|d_h \Pi_h \alpha - \Pi_h d\alpha\|_{L^2} \leq C h^p \|\alpha\|_{H^{p+1}}.$$

Lemma 5.4 (Codifferential consistency)**.** *For smooth* α,

$$\|\delta_h \Pi_h \alpha - \Pi_h \delta \alpha\|_{L^2} \leq C h^p \|\alpha\|_{H^{p+1}}.$$

5.1 Gauge and Hodge decomposition

Any discrete 1-form admits the decomposition

$$A_h = d_h \phi_h \oplus \delta_h^* \psi_h \oplus h_h.$$

Imposing $\delta_h A_h = 0$ enforces a discrete Lorenz gauge, which converges to the continuous condition $\delta A = 0$.

6 Error Constants

For linear elements ($p = 1$), constants can be expressed in terms of the mesh aspect ratio κ:

$$\begin{aligned} c_1 &\geq \frac{1}{1+\kappa}, \\ c_2 &\leq 1 + \kappa, \\ C &\leq 1 + \kappa. \end{aligned}$$

Constant	Meaning	Dependence	Example (3+1D)
c_1	Norm equivalence (DEC)	Mesh aspect ratio	0.8–1.2
c_2	FEEC stability	Polynomial degree p	≈ 1.5 for $p = 1$
C	Error bound constant	$\max(c_1^{-1}, c_2)$	2.0

Table 1: Summary of constants for 3+1D Maxwell system.

7 Figures

Nested mesh with UNNS potentials

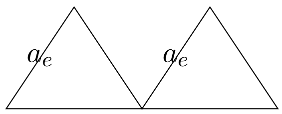

Figure 1: Schematic nested mesh with UNNS-assigned edge values.

$$A \text{ (1-forms)} \longrightarrow F = dA \text{ (2-forms)} \longrightarrow J \text{ (currents)}$$

Field tower in spacetime

Figure 2: Field tower: potential → field → current.

8 Discussion

UNNS ensures refinement $\implies$ mesh family exists. DEC ensures exactness, FEEC ensures stability. Thus UNNS provides a constructive number-theoretic foundation for electromagnetism: discrete numbers at each mesh level converge to smooth fields, with constants and convergence rates controlled by FEEC theory.

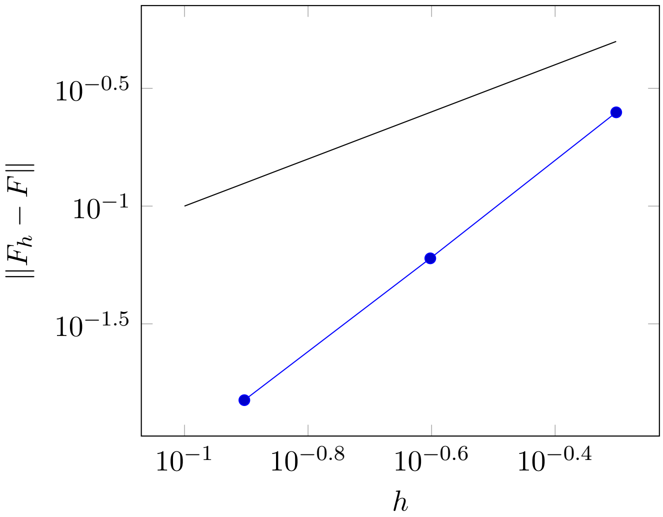

Figure 3: Convergence plot $\|F_h - F\| \sim h^p$ with $p = 1$.

References

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