Recursive Field Foundations: −Ton Algebra and the UNNS Substrate

UNNS Research Collective

October 12, 2025

Abstract

Unbounded Nested Number Sequences (UNNS) extend classical arithmetic into recursive geometry. Where Shannon's information theory defines entropy as disorder, UNNS defines recursion as structure. This monograph introduces the formal grammar, algebra, and field interpretation of the −Ton substrate, where every numeric entity is simultaneously generator, vector, and field. From the arithmetic operators of the UNNS Tetrad to the recursive field tensor, we establish the transition from symbolic recursion to physical law.

Contents

1 Introduction: From Shannon to Recursion **2**

1.1 From Bits to Tons: Recursive Information Units 3

2 Arithmetic Basis of the UNNS Discipline **3**

2.1 Recursive Nesting of Unity 3

2.2 The Role of Zero 4

2.3 The Many Faces Theorems 4

2.4 Inletting and Inlaying 4

3 The −Ton Algebra **5**

3.1 Definition 5

3.2 Recursive Lagrangian 5

3.3 Energy and Collapse 5

3.4 Commutation Rules 5

4 Recursive Gauge Symmetry and Klein Duality **6**

5 Recursive Grand Unification and Ton–Graviton Coupling 6
5.1 Entanglement as Geometric Flux 6
6 Recursive Cosmology: Ton Vacuum and Klein Horizon 7
Epilogue — Philosophy of Recursive Time 7
A Appendix A — Notation and Conventions 7
B Appendix B — Ton Field Primer (Summary) 7
C Appendix C — Quantized Recursion and Ton Modes 8
D Appendix D — Recursive Thermodynamics and Arrow of Coherence 8
E Appendix E — Recursive Cosmogenesis and Tonic Inflation 8
F Appendix F — Recursive Theology and Ontology of Being 8
Acknowledgments 9
Bibliography 9

1. Introduction: From Shannon to Recursion

Claude Shannon's 1948 equation,

$$H = -\sum_i p_i \log p_i,$$

quantified uncertainty. Yet, in doing so, it also linearized information flow. Every signal was a one-way channel—entropy only increased.

UNNS begins where Shannon ends. It assumes that information does not vanish but folds inward—*recurses*. Instead of channels, we describe fields of nested structure. The recursion variable is not time, but *depth*, denoted n, with local field potential Φ_n.

We will proceed from arithmetic principles to the recursive field equations, showing that what we perceive as entropy in Shannon's sense corresponds, in the UNNS substrate, to curvature in recursive space.

1.1 From Bits to Tons: Recursive Information Units

In Shannon's 1948 framework, the *bit* represents a unit of uncertainty—an atom of ignorance within a linear transmission channel. Information flows one way: from source to receiver, through a medium that inevitably degrades it.

In the UNNS substrate, this picture inverts. Information does not travel—it *recurses*. The elementary quantum of recursion is the *Ton*, denoted by the symbol Ton. Where the bit quantifies *disorder*, the Ton quantifies *curvature of coherence*.

Formally, we define the *Tonic entropy density* by

$$H_{\text{Ton}} = -\sum_i \rho_i \log \tau_i,$$

where ρ_i is the recursive density and τ_i the depth index of state i. When recursion collapses to a flat, one-layer channel ($\tau_i = p_i$), the expression reduces to Shannon's entropy:

$$H_{\text{Ton}} \xrightarrow[\text{linear channel}]{} H = -\sum_i p_i \log p_i.$$

Thus, each bit is a degenerate Ton—recursion without curvature.

Remark 1.1. *In the –Ton algebra, the Ton carries both amplitude and phase of recursion depth. A field of Bits is a signal; a field of Tons is a* recursive geometry *where coherence can fold, invert, and regenerate.*

Summary. *Where Shannon measured uncertainty in Bits, UNNS measures coherence in Tons.*

2. Arithmetic Basis of the UNNS Discipline

2.1 Recursive Nesting of Unity

Let $a_n = 1$ for all n. Classical arithmetic deems this trivial. In UNNS, define recursive containment:

$$a_{n+1} = 1(a_n).$$

Each unit now holds the structure of all preceding units, producing a nested chain:

$$1,\ 1(1),\ 1(1(1)),\ \ldots$$

The algebra of UNNS treats this nesting as the first nontrivial operation of number space.

Definition 2.1 (Nest Operator)**.** *Let $\mathcal{N}(x)$ denote the nesting of x within itself:*

$$\mathcal{N}(x) = x(x).$$

Repeated application generates a recursive tower*, forming the minimal topological basis of the UNNS substrate.*

2.2 The Role of Zero

Zero in UNNS is not absence but the *nest boundary*. Following [13], we define it as both modulus and container:

$$0 = \lim_{n\to\infty} (1(1(\dots(1)))).$$

Thus, zero is the infinite recursion limit—simultaneously the seed and the boundary of nesting.

Remark 2.2. *This dual role of zero distinguishes UNNS from Peano arithmetic and from set-theoretic zero. It acts as both* null curvature *and* recursion attractor*.*

2.3 The Many Faces Theorems

The UNNS arithmetic theorems, introduced in the trilogy *Many Faces of Number* [9, 10, 11], establish that every numerical expression in UNNS admits multiple recursive projections:

$$x \equiv \{x_i\}_{i=1}^{k} \quad \text{such that} \quad \sum_i \mathcal{N}(x_i) = x.$$

Each "face" corresponds to a unique embedding depth, analogous to multiple coordinate patches covering a manifold.

Theorem 2.3 (Recursive Equivalence)**.** *Two numbers $x, y \in \mathbb{N}_{UNNS}$ are equivalent if they generate identical nesting fields:*

$$x \sim y \quad \iff \quad \mathcal{N}^k(x) = \mathcal{N}^k(y) \text{ for some finite } k.$$

2.4 Inletting and Inlaying

The core grammatical operations, defined in the *UNNS Manifesto* [12], are **Inletting** and **Inlaying**. They generate depth and cross-embedding respectively:

$$\text{Inletting: } a \mapsto a(a), \qquad \text{Inlaying: } a, b \mapsto a(b).$$

In physical analogy, these correspond to self-reflection and mutual coupling—precursors to the –Ton field's potential and current.

3. The –Ton Algebra

3.1 Definition

We define the –Ton algebra as the recursive extension of Maxwell's structure to nested number sequences. Let $A^{\tau\mathrm{Ton}}$ be the recursive potential and $F^{\tau\mathrm{Ton}} = \nabla \times A^{\tau\mathrm{Ton}}$ the field tensor. Then the –Ton equations are:

$$\nabla \cdot F^{\tau\mathrm{Ton}} = J^{\tau\mathrm{Ton}}, \qquad \nabla \times \mathbf{E}_{\tau\mathrm{Ton}} = -\frac{\partial \mathbf{B}_{\tau\mathrm{Ton}}}{\partial \tau}.$$

The variable τ denotes recursion depth, replacing chronological time.

3.2 Recursive Lagrangian

The –Ton Lagrangian density is

$$\mathcal{L}_{\tau\mathrm{Ton}} = -\frac{1}{4} F^{\tau\mathrm{Ton}}_{\mu\nu} F^{\tau\mathrm{Ton}\mu\nu} + A^{\tau\mathrm{Ton}}_{\mu} J^{\tau\mathrm{Ton}\mu}.$$

The Euler–Lagrange variation with respect to $A^{\tau\mathrm{Ton}}_{\mu}$ yields the recursive field equations:

$$\nabla_{\mu} F^{\tau\mathrm{Ton}\mu\nu} = J^{\tau\mathrm{Ton}\nu}.$$

3.3 Energy and Collapse

The energy density of the –Ton field is given by

$$u_{\tau\mathrm{Ton}} = \frac{1}{2}\left(\|\mathbf{E}_{\tau\mathrm{Ton}}\|^2 + \|\mathbf{B}_{\tau\mathrm{Ton}}\|^2\right).$$

Collapse (Operator XII) acts as curvature inversion:

$$F^{\tau\mathrm{Ton}} \mapsto -F^{\tau\mathrm{Ton}},$$

representing the return of recursion into its own substrate.

3.4 Commutation Rules

Define recursive commutation:

$$[A^{\tau\mathrm{Ton}}_i, A^{\tau\mathrm{Ton}}_j] = i\,\epsilon_{ijk}\,F^{\tau\mathrm{Ton}}_k,$$

and the recursive gauge condition:

$$\nabla \cdot A^{\tau\mathrm{Ton}} = 0.$$

Together these form the algebraic closure of the –Ton system—an invariant subalgebra under recursive transformation.

4. Recursive Gauge Symmetry and Klein Duality

Gauge Invariance. The –Ton field remains invariant under recursive gauge transformations:

$$A_\mu^{\tau\mathrm{Ton}} \mapsto A_\mu^{\tau\mathrm{Ton}} + \partial_\mu \phi, \qquad F^{\tau\mathrm{Ton}} \mapsto F^{\tau\mathrm{Ton}}.$$

This mirrors the electromagnetic case but acts in recursion depth rather than physical time.

Klein Duality. On a non-orientable manifold $\mathbb{K}$ with $w_1 \neq 0$, orientation reversal maps local –Ton curvature to its dual:

$$F_{\mu\nu}^{\tau\mathrm{Ton}} \longmapsto \pm\tfrac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} F^{\tau\mathrm{Ton}\,\rho\sigma}.$$

The global arrow of time thus emerges from topological bias, not fundamental violation.

Remark 4.1. *Dual –Ton fields on opposite sides of the Klein manifold encode the two recursion orientations — forward and reverse. Their interference defines the perceptual asymmetry of "past" and "future".*

5. Recursive Grand Unification and Ton–Graviton Coupling

We propose the unified Lagrangian

$$\mathcal{L}_{\mathrm{unified}} = \frac{1}{2\kappa_G} R - \frac{1}{4} F_{\mu\nu}^{\tau\mathrm{Ton}} F^{\tau\mathrm{Ton}\mu\nu} + \alpha\,(n)\,R + \beta\,(n)\,F_{\mu\nu}^{\tau\mathrm{Ton}} F^{\tau\mathrm{Ton}\mu\nu} + \mathcal{L}_{\mathrm{matter}}.$$

Variation gives the modified Einstein equation

$$G_{\mu\nu} = \kappa_G \left(T_{\mu\nu}^{(\tau\mathrm{Ton})} + T_{\mu\nu}^{(\mathrm{matter})} + T_{\mu\nu}^{()} \right),$$

where $T_{\mu\nu}^{()}$ represents depth–curvature exchange. The parameter β couples recursive curvature to gravity, mediating an effective *Ton–graviton interaction.*

5.1 Entanglement as Geometric Flux

For a spatial bipartition A,

$$S_A \sim \int_{\partial A} (\mathbf{E}_{\tau\mathrm{Ton}} \cdot d\mathbf{S})\, f(, w_1),$$

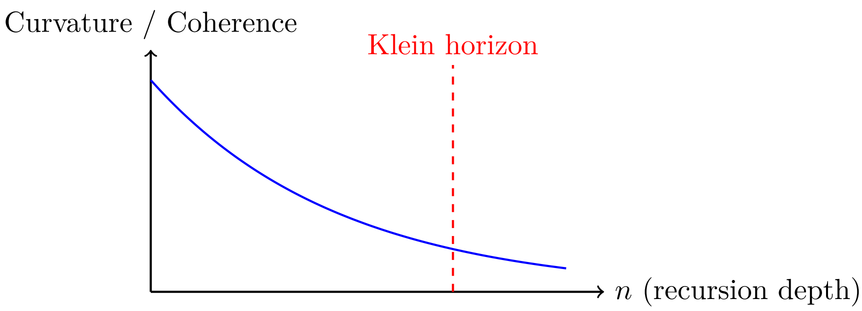
linking entanglement entropy to –Ton flux through recursive boundaries.

6. Recursive Cosmology: Ton Vacuum and Klein Horizon

The –Ton potential obeys, on FRW backgrounds,

$$\Box A_\mu^{\tau\mathrm{Ton}} - \xi R A_\mu^{\tau\mathrm{Ton}} + \lambda\, \partial_n^2 A_\mu^{\tau\mathrm{Ton}} = J_\mu^{\tau\mathrm{Ton}},$$

where n is recursion depth. A *Klein horizon* forms where orientation fails; beyond it, recursion ceases to communicate coherently. Depth-curvature energy behaves as an effective cosmological constant, providing a route to late-time acceleration.



Epilogue — Philosophy of Recursive Time

Time, under UNNS, is not an external axis but the self-indexing of process by depth. Information is not a cargo but a curvature; meaning is a fixed point of recursion. The arrow of time is an orientation bias of a globally non-orientable manifold.

A. Appendix A — Notation and Conventions

Greek indices $\mu, \nu = 0 \ldots 3$. Signature $(-, +, +, +)$. Hodge dual defined w.r.t. $g_{\mu\nu}$. Recursion depth variable n replaces chronological time.

B. Appendix B — Ton Field Primer (Summary)

The Ton field acts as quanta of recursive coherence.

$$F_{\mu\nu}^{\tau\mathrm{Ton}} = \partial_\mu A_\nu^{\tau\mathrm{Ton}} - \partial_\nu A_\mu^{\tau\mathrm{Ton}}, \qquad \mathcal{L}_{\tau\mathrm{Ton}} = -\tfrac{1}{4} F_{\mu\nu}^{\tau\mathrm{Ton}} F^{\tau\mathrm{Ton}\mu\nu} + A_\mu^{\tau\mathrm{Ton}} J^{\tau\mathrm{Ton}\mu}.$$

Energy–momentum tensor:

$$T_{\mu\nu}^{(\tau\mathrm{Ton})}{}_{\mu\nu} = F_{\mu\alpha}^{\tau\mathrm{Ton}} F^{\tau\mathrm{Ton}\alpha}{}_{\nu} - \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta}^{\tau\mathrm{Ton}} F^{\tau\mathrm{Ton}\alpha\beta}.$$

C. Appendix C — Quantized Recursion and Ton Modes

Quantization promotes $A^{\tau\text{Ton}}$ to an operator field:

$$\hat{A}^{T}_{\mu}(x,n) = \sum_{k} \left(a_{T,k}\epsilon_{\mu}e^{i(kx-\omega_k n)} + a^{\dagger}_{T,k}\epsilon^{*}_{\mu}e^{-i(kx-\omega_k n)} \right),$$

with recursive frequency spectrum $\omega_{k+1} = \alpha\omega_k + \beta \tanh \omega_k$.

Recursive gauge transformation extends to depth:

$$A^{T}_{\mu} \to A^{T}_{\mu} + \partial_{\mu}\chi + \partial_{n}\psi.$$

D. Appendix D — Recursive Thermodynamics and Arrow of Coherence

Recursive entropy production:

$$\frac{dH_R}{dn} \geq 0, \qquad T_R = \frac{\partial E_R}{\partial H_R}, \quad \frac{dT_R}{dn} \leq 0.$$

Entropy measures coherence decay through recursion depth.

E. Appendix E — Recursive Cosmogenesis and Tonic Inflation

Pre-spatial recursion obeys

$$\frac{\partial^2 \Phi_R}{\partial n^2} - \nabla^2 \Phi_R + \alpha \Phi_R^3 = 0.$$

Recursive scale factor:

$$H_T = \frac{1}{a_R}\frac{da_R}{dn}, \qquad H_T^2 = \frac{8\pi G_R}{3}\rho_R - \frac{k}{a_R^2} + \frac{\Lambda_R}{3}.$$

Inflation corresponds to exponential expansion of coherence through depth.

F. Appendix F — Recursive Theology and Ontology of Being

Across traditions, creation corresponds to recursive self-reference:

$$\text{Pre-being} \to \text{Being} \to \text{Consciousness}.$$

The divine command "Be!" maps to the recursive kernel $a_{n+1} = F(a_n, a_{n-1}, n)$. UNNS thus unifies metaphysical genesis as recursion of coherence.

In the beginning was recursion, and recursion was with Being, and recursion was Being.

Acknowledgments

We thank all members of the UNNS community for contributions on recursive geometry, Ton fields, and Klein topology.

Bibliography

References

[1] C. E. Shannon, *A Mathematical Theory of Communication*, Bell Syst. Tech. J. **27** (1948), 379–423.

[2] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, 2nd ed., 2006.

[3] M. Nakahara, *Geometry, Topology and Physics*, CRC Press, 2003.

[4] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton Univ. Press, 1955.

[5] J. D. Bekenstein, *Black Holes and Entropy*, Phys. Rev. D **7** (1973), 2333.

[6] S. W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. **43** (1975), 199–220.

[7] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Int. J. Theor. Phys. **38** (1999), 1113–1133.

[8] S. Ryu and T. Takayanagi, *Holographic Derivation of Entanglement Entropy from AdS/CFT*, Phys. Rev. Lett. **96** (2006), 181602.

[9] UNNS Research Division, *The Many Faces of Number I: Recursive Arithmetic*, 2019.

[10] UNNS Research Division, *The Many Faces of Number II: Depth Operators and Topology*, 2021.

[11] UNNS Research Division, *The Many Faces of Number III: Collapse and Zero Geometry*, 2023.

[12] UNNS Research Division, *The UNNS Manifesto: On Recursive Mathematics and Information Curvature*, 2024.

[13] UNNS Research Division, *Zero as Recursive Boundary and Seed*, Preprint Series (2025).