A UNNS Critique of Banach–Tarski and the Axiom of Choice

UNNS Research Notes

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Abstract

This note explores how the Unbounded Nested Number Sequence (UNNS) framework offers a constructive alternative to classical set-theoretic models based on the real continuum and the Axiom of Choice (AC). We argue that paradoxes such as Banach–Tarski are artifacts of infinitely divisible space combined with non-constructive choice. Within UNNS, only constructive, recursively generated nests exist, and this removes the logical space for non-measurable sets.

1 Background

Classical mathematics models the physical world with real numbers $\mathbb{R}$. This provides a smooth continuum useful for analysis and physics, but its reliance on infinite divisibility and AC enables counterintuitive paradoxes such as the Banach–Tarski (B–T) decomposition: a solid ball in $\mathbb{R}^3$ can be partitioned into finitely many pieces and reassembled into two balls of the same size.

Most mathematicians accept B–T as a theorem; however, from a UNNS perspective, it highlights a misalignment between physical recursion and set-theoretic abstraction.

2 UNNS Principles

The UNNS substrate is governed by three guiding principles:

1. **Recurrence sufficiency:** Every constructible object arises from initial seeds and recursive generation. Arbitrary selections without recursive basis are disallowed.
2. **Nested discreteness:** Objects live in nested lattices, not in a continuum. Infinite divisibility of space is not assumed.
3. **Operational grammar:** All constructions are performed by explicit operators (inletting, inlaying, repair, projection, etc.).

3 Non-measurable sets under UNNS

In classical measure theory, AC implies the existence of non-measurable sets. These are essential for the B–T paradox.

Lemma 3.1 (No non-measurable sets under UNNS recursion)**.** *Within the UNNS framework, every admissible set is generated by a finite or countably recursive process. Consequently, each admissible set carries a canonical measure induced by its recursive structure. Therefore, non-measurable sets cannot arise.*

Proof. By recurrence sufficiency, any set S is either finite (measurable by counting) or countably recursive (measurable via a recursive sigma-algebra). Since arbitrary choice selections are disallowed, no Vitali-like construction is possible. Thus, all sets generated by UNNS are measurable. □

Remark 3.2. *This lemma shows that B–T decompositions, which rely on non-measurable pieces, are impossible in UNNS. Recursion preserves measurability.*

4 Axiom of Choice in UNNS

AC allows arbitrary selection from arbitrary sets. In UNNS, selection must be constructive:

Proposition 4.1 (Replacement of AC by Recurrence)**.** *In UNNS, the role of AC is replaced by* recurrence sufficiency*: all admissible sets are explicitly generated, so no external choice principle is required.*

5 Discussion

The UNNS perspective reframes paradoxes:

- B–T is not a "truth" but a sign of model breakdown.
- $\mathbb{R}$ is not fundamental; it is a continuum approximation of nested recursion.
- AC is too permissive; UNNS enforces constructive sufficiency.

6 Conclusion

Through the UNNS lens, Banach–Tarski and the Axiom of Choice are seen not as necessary features of mathematics but as artefacts of unsuitable modeling assumptions. UNNS offers a constructive, measurable, and recursion-grounded substrate, better aligned with both computation and physical reality.